The Method of Direct Proof

(Used Only for Proving Universal Statements)

The Design for Direct Proofs of Universal Statements of the form:

$\forall x \in D$, predicate $P(x)$.

To Prove: $\forall\ x \in D$, $P(x)$.

Proof: Let x be any element of the domain D.
[N. T. S. $P(x)$ is true about this particular x value.]
. . .
. . .
$\therefore P(x)$.
$\therefore$ For every element x in D, $P(x)$, by Direct Proof. QED [*quod erat demonstrandum*]

For a proof of a universal conditional statement, the proof requires a particular extended form:

The Design for Direct Proofs of Universal IF-THEN Statements:

$\forall x \in D$, IF $P(x)$, THEN $Q(x)$.

To Prove: $\forall\ x \in D$, IF $P(x)$, THEN $Q(x)$.

Proof: Let x be any element of the domain D. [First, define your variables!]

Suppose $P(x)$. [Suppose the "IF" part !]

[N. T. S. $Q(x)$ is true about this particular x value.]
. . .
. . .
. . .

$\therefore Q(x)$. [Conclude the "THEN" part, proving "If $P(x)$, Then $Q(x)$" about the generic value of x.]

$\therefore$ For every element x in D, IF $P(x)$, THEN $Q(x)$, by Direct Proof.

QED

Note: The two statements "Let x be any ..." and "Suppose P(x)"

can be combined in the single statement:

"Let x be any element of the domain D such that $P(x)$."

As an example, the beginning on a proof of the statement, "For all integers n, if n is odd, then $n^2 + 1$ is even," could begin as follows (and in either form):

"Let n be an integer.
Suppose that n is an odd number.
[NTS: $n^2 + 1$ is even]"

"Let n be an integer such that n is odd".
[NTS: $n^2 + 1$ is even]"

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(Used Only for Proving Universal Statements)

The method of "Direct Proof" is used only for proving a statement which is quantified by a universal quantifier, e.g., "For all . . . ," "For every . . . ," etc.

Two other proof methods (Mathematical Induction and Proof-by-Contradiction) can also be used for the proof of a universal statement, but the method of Direct Proof can only be used in proving a universal statement.

If the statement-to-prove is not a universal statement,
then DO NOT WRITE "by Direct Proof" at the end of the proof.

Definition of "Direct Proof Method": A proof using the "Direct Proof Method" is one in which a chosen variable represents an arbitrary element of the domain and the predicate of the universal statement is proved to be true about that arbitrary element.

The simplest form of a universal statement (in symbolic terms) is: $\forall x \in D$ **, predicate P(x) .**

[Note: D is the domain of the variable x and P(x) is a predicate which makes an assertion about the entity that the variable x represents, e.g., "For every integer x, $x^2 \geq 0$," in which $D = \mathbb{Z}$ and $P(x) =$ " $x^2 \geq 0$."]

The Steps in the Method of Direct Proof

1) Begin with a statement which defines a variable, (say, for example, x,) which represents a particular but arbitrarily chosen element of the domain D.
2) Operating with expressions in terms of x, make a logical argument which concludes essentially,
"Therefore, P(x) is true."
(This statement is referring only to the particular but arbitrarily chosen value of x, selected at the start.)
3) When the truth of P(x) has been established regarding the particular but arbitrarily chosen value of x, then one can conclude that the P(x) is true **for all** values of x, and one writes:
" Therefore, for all $x \in D$, predicate P(x) is true , by Direct Proof."

[Note: The most common form of the predicate P(x) is that of a conditional (If-Then) statement, "$A \rightarrow B$" ;
thus, often the form of the universal statement to be proved is: $\forall x \in D$ **, P(x) → Q(x) .**
When this is the case, the definition of the generic particular variable x is followed by the supposition that the "IF" part of the conditional statement is true, that is, by "Suppose P(x)."
This should be followed by the comment [NTS Q(x)], where NTS means "need to show".]

When the conclusion "Therefore, Q(x)" is achieved, the conditional "If P(x), Then Q(x)" has been proved, and a conclusion of the universal statement ends the proof.

[Note: The defining of the particular but arbitrarily chosen element x from domain D can be accomplished with one of three wordings:

1) "Let x be any . . . (such and such)", as in "Let x be any integer."
2) "Let (such and such) x be given", as in "Let integer x **be given**" or "Let $x \in \mathbb{Z}$ be given",

(writing "Let $x \in \mathbb{Z}$", without the "be given", is incorrect.)]

A Good Proof and a Bad Proof

A Good Proof:

To Prove: The sum of an even integer and an odd integer is odd.

[The form in symbols: $\forall\, m \in \mathbb{Z}, \forall\, n \in \mathbb{Z}$, if m is even and n is odd, then $(m + n)$ is odd.]

Proof: Let m and n be any integers.

Suppose m is even and n is odd. [NTS: $m + n$ is odd.]

$\therefore$ By the definitions of "even" and "odd", there exist integers s and t such that $m = 2s$ and $n = 2t + 1$.

$\therefore m + n = 2s + (2t + 1)$, by substitution,

$= 2(s + t) + 1$, by Rules of Algebra.

Let $k = s + t$, which is an integer since sums and products of integers are integers.

So, $m + n = 2k + 1$, by substitution.

$\therefore m + n$ is odd, by definition of "odd".

$\therefore$ The sum of an even integer and an odd integer is an odd integer, by Direct Proof. QED

Good Points about the Good Proof:

"Let" has been used for the initial definition of the variables (rather than "Suppose").

"Suppose" has been used for further restricting the domains of the variables.

Every deduction is justified with a reason.

Every comment is enclosed in brackets: [C O M M E N T]

[A "Comment" is a statement which does not advance the proof but which only aids the reader (and often the writer too) in understanding what is going on in the proof.]

The statement-to-prove is re-formulated as an equivalent "IF-Then" statement at the start [and in a comment] to aid the reader (and the proof writer, too).

The calculation $(s + t)$,
which is the integer required by the definition of "odd" to prove that $m + n$ is odd,
is used to define a third variable, k, so that the proof includes the statement,

$\therefore m + n = 2k + 1$, rather than the statement,

$\therefore m + n = 2$ (some integer) $+ 1$, which is NOT ALLOWED, but which you will see in the book.

A Bad Proof:

To Prove: The sum of an even integer and an odd integer is odd.

Proof: Suppose $2s$ and $2t+1$ are arbitrarily chosen integers.

$$(2s)+(2t+1) = 2(s+t)+1$$

$$\Rightarrow$$

$$(2s)+(2t+1) = 2\,(\text{some integer})+1.$$

∴ The sum of an even integer and an odd integer is an odd integer, by Direct P:roof. QED

Bad Points about the Bad Proof:

s and t are not adequately defined as representing integers.

You must separately define m and n as an even integer and odd integer, respectively, in one step, and then, **in another step**, apply the definitions of "even" and "odd" to define integers s and t .

The "some integer" factor is not allowed as discussed above.

The use of logical symbols, such as ∃ , ∀ , ~ , → , ⇔ , ⇒ , are not allowed in proof statements

(but they may be used in COMMENTS, which are placed in brackets [see directly below]).

The use of symbols, including logical symbols, such as ∃ , ∀ , ~ , → , ⇔ , ⇒ , are allowed in proofs written in words when they are used in COMMENTS . [COMMENTS must always be placed in brackets!]

Also, **do not use the word "If" when the meaning is "Because". Use "Because" or "Since".**

Ex: INCORRECT! → "Let n be an even number. If n is even, then n = 2k for some integer k ."

CORRECT! → "Let n be an even number. Because n is even, n = 2k for some integer k ."